Dada la serie:

$$S = 1 + 2^p x + 3^p x^2 + \cdots + n^p x^{n-1}, \quad p > 0$$

Luego,

$$S = \sum_{k=0}^{n-1} (k+1)^p \, x^k$$

$$S - xS = \sum_{k=0}^{n-1} (k+1)^p \, x^k - \sum_{k=0}^{n-1} (k+1)^p \, x^{k+1}$$

$$= \sum_{k=0}^{n-1} (k+1)^p \, x^k - \left(\sum_{k=0}^{n-2} (k+1)^p \, x^{k+1} + n^p x^n \right)$$

$$= \sum_{k=0}^{n-1} (k+1)^p \, x^k - \left(\sum_{k=1}^{n-1} k^p \, x^k + n^p x^n \right)$$

$$= \left(\sum_{k=0}^{n-1} (k+1)^p \, x^k - \sum_{k=0}^{n-1} k^p \, x^k \right) - n^p x^n$$

$$= \left(\sum_{k=0}^{n-1} (k+1)^p - k^p \right) x^k - n^p x^n$$

$$= \sum_{k=0}^{n-1} \left(\sum_{j=0}^{p-1} \binom{p}{j} k^j \right) x^k - n^p x^n$$

$$(1 - x)S = \sum_{j=0}^{p-1} \binom{p}{j} \left(\sum_{k=0}^{n-1} k^j x^k \right) - n^p x^n$$

Observación:

$$\sum_{k=0}^{n} \left(\sum_{j=0}^{p} v_{p,j} w_{k,j} \right) u_k = \sum_{j=0}^{p} v_{p,j} \left(\sum_{k=0}^{n} u_k w_{k,j} \right)$$

Por consiguiente,

$$S = \frac{1}{1 - x} \left(\sum_{j=0}^{p-1} \binom{p}{j} \left(\sum_{k=0}^{n-1} k^j \, x^k \right) - n^p x^n \right)$$

Donde,

$$\sum_{k=0}^{n-1} x^k = \frac{1 - x^n}{1 - x}$$